Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

A botanist is conducting an experiment which studies the growth of plants.

The heights of the plants are measured on seven different days.

The following table shows the number of days, d, that the experiment has been running and the average height, $h\,\text{cm}$, of the plants on each of those days.

Number of days (d)	2	5	13	24	33	37	42
Average height (h)	10	16	30	59	76	79	82

The value of Pearson's product-moment correlation coefficient, r, for this data is 0.991, correct to three significant figures.

(a) The regression line of h on d for this data can be written in the form $h = ad + b$.

Find the value of a and the value of b. [2]

(b) Use your regression line to estimate the average height of the plants when the experiment has been running for 20 days. [2]

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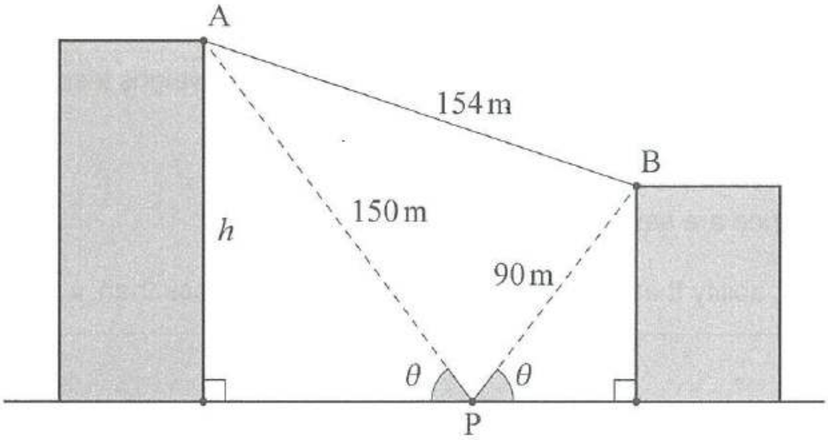
2. [Maximum mark: 6]

The following diagram shows two buildings situated on level ground.

From point P on the ground directly between the two buildings, the angle of elevation to the top of each building is θ.

diagram not to scale



The distance from point P to point A at the top of the taller building is 150 metres.

The distance from point P to point B at the top of the shorter building is 90 metres.

The distance between A and B is 154 metres.

Find the height, h, of the taller building.

3. [Maximum mark: 8]

The weights, W grams, of bags of rice packaged in a factory can be modelled by a normal distribution with mean 204 grams and standard deviation 5 grams.

(a) A bag of rice is selected at random.

Find the probability that it weighs more than 210 grams. [2]

According to this model, 80% of the bags of rice weigh between w grams and 210 grams.

(b) Find the probability that a randomly selected bag of rice weighs less than w grams. [2]

(c) Find the value of w. [2]

(d) Ten bags of rice are selected at random.

Find the probability that exactly one of the bags weighs less than w grams. [2]

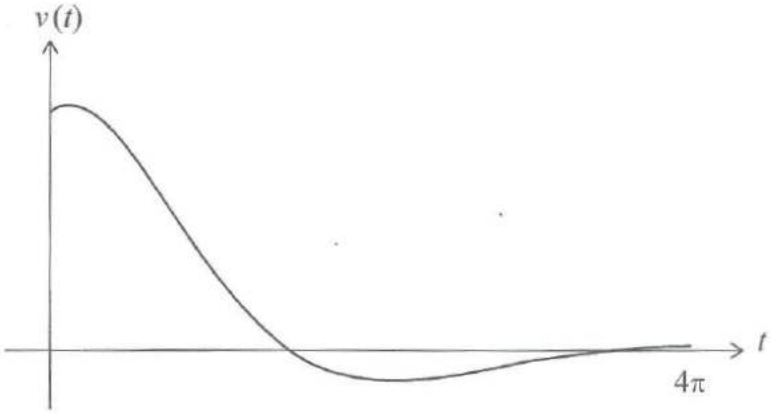
4. [Maximum mark: 7]

The expansion of $(x+h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

Given that the coefficients, a, b and d, are the first three terms of a geometric sequence, find the value of h.

5. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of v is shown in the following diagram.



Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1. [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2. [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

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6. [Maximum mark: 8]

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let L be the line of intersection of Π_1 and Π_2.

(a) Verify that a vector equation of L is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]

(b) Find the coordinates of the point P on L that is nearest to the origin. [5]

7. [Maximum mark: 5]

A function f is defined as $f(x) = \arctan(x-2)$, where $2 \le x \le 2+\sqrt{3}$.

The region bounded by the curve, the y-axis, the x-axis and the line $y = \frac{\pi}{3}$ is rotated 360° about the y-axis to form a solid of revolution.

Find the volume of the solid.

8. [Maximum mark: 7]

A function g is defined by $g(x) = \frac{2x-5}{x^2-3}$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$.

(a) Determine the range of g. [4]

A function h is defined by $h(x) = g(|x|)\cos t$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$ and t is a constant where $\frac{\pi}{2} < t \leq \pi$.

(b) Find the set of values of x such that $h(x) \leq 0$. [3]

9. [Maximum mark: 5]

Let S be the set of 30 positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from S without replacement. He then adds them together and determines whether the sum is divisible by 3.

Determine the total number of ways Raghu can obtain a sum that is divisible by 3. [5]

Do **not** write solutions on this page.

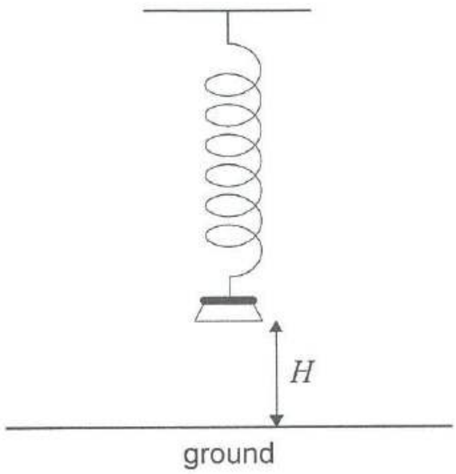
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 13]

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

The height, H metres, of the base of the weight above the ground can be modelled by the function $H(t) = a\cos(7.8t) + b$, for a, $b \in \mathbb{R}$ and $0 \le t \le 10$, where t is the time in seconds after the weight is released.



(a) Find the period of the function. [2]

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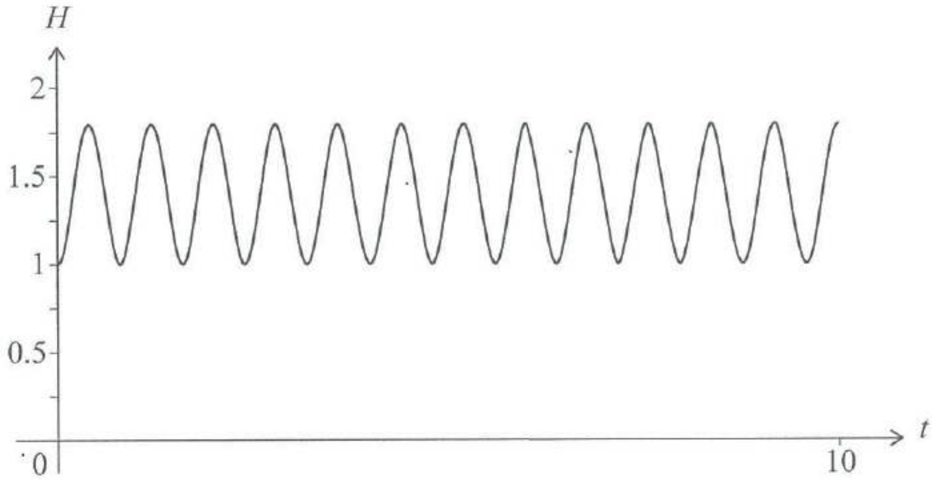
Do **not** write solutions on this page.

(Question 10 continued)

The weight is released when its base is at a minimum height of 1 metre above the ground, and it reaches a maximum height of 1.8 metres above the ground. The graph of H is shown in the following diagram.



(b) Find the value of

(i) a;

(ii) b. [3]

(c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]

(d) Find the first time that the base of the weight reaches a height of 1.5 metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

(e) Find the probability that the height of the base of the weight is greater than 1.5 metres at the time the picture is taken. [4]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

A game of chance involves drawing **two** balls at random out of a box without replacement. The box initially contains r red balls and y yellow balls.

Let $\mathrm{P}(YY)$ represent the probability of drawing two yellow balls from the box without replacement.

Consider a version of this game where it is known that $\mathrm{P}(YY)=\frac{1}{3}$.

(a) Show that $2y^2-2(r+1)y+r-r^2=0$. [4]

(b) By solving the equation in part (a), show that $y=\frac{(r+1)+\sqrt{3r^2+1}}{2}$. [4]

(c) Find two pairs of values for r and y that satisfy the condition $\mathrm{P}(YY)=\frac{1}{3}$. [4]

Now consider a similar game of chance that involves drawing **three** balls out of a box without replacement. The box initially contains 10 red balls and y yellow balls.

Let $\mathrm{P}(YYY)$ represent the probability of drawing three yellow balls from the box without replacement.

(d) Find an expression for $\mathrm{P}(YYY)$ in terms of y. [3]

A yellow ball is added so that the box now contains 10 red balls and $(y+1)$ yellow balls. The probability of drawing three yellow balls from the box without replacement is now twice the probability expressed in part (d).

(e) Find the initial number of yellow balls in the box. [5]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0$, $y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of y when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \le x \le 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y, determine the x-coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0$, $y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y = mx$ where the exact value of m is to be determined. [6]